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Research Article

Robust Stability Analysis of Fuzzy Neural Network with Delays

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We investigate local robust stability of fuzzy neural networks (FNNs) with time-varying and S-type distributed delays. We derive some sufficient conditions for local robust stability of equilibrium points and estimate attracting domains of equilibrium points except unstable equilibrium points. Our results not only show local robust stability of equilibrium points but also allow much broader application for fuzzy neural network with or without delays. An example is given to illustrate the effectiveness of our results.

1. Introduction

For the study of current neural network, two basic mathematical models are commonly adopted: either local field neural network models or static neural network models. The basic model of local field neural network is described as

$$\dot{x}_i(t) = -x_i(t) + \sum_{j=1}^{n} \omega_{ij} g_j\left(x_j(t)\right) + I_i, \quad i = 1, 2, \dots, n, \tag{1.1}$$

where g_j denotes the activation function of the jth neuron; x_i is the state of the ith neuron; I_i is the external input imposed on the ith neuron; ω_{ij} denotes the synaptic connectivity value between the ith neuron and the jth neuron; n is the number of neurons in the network. With the same notations, static neural network models can be written as

$$\dot{x}_i(t) = -x_i(t) + g_i\left(\sum_{j=1}^{n} \omega_{ij} x_j(t) + I_i\right), \quad i = 1, 2, \dots, n. \tag{1.2}$$

It is well known that local field neural network not only models Hopfield-type networks [1] but also models bidirectional associative memory networks [2] and cellular neural networks [3]. Many deep theoretical results have been obtained for local field neural network; we can refer to [4–12] and references cited therein. Meanwhile static neural network has a great potential of applications. It not only includes the recurrent back-propagation network [13–15] but also includes other extensively studied neural network such as the optimization type network introduced in [16–18] and the brain-state-in-a-box (BSB) type network [19, 20]. In the past few years, there has been increasing interest in studying dynamical characteristics such as stability, persistence, periodicity, local robust stability of equilibrium points, and domains of attraction of local field neural network (see[21–25]).

However, in mathematical modeling of real world problems, we will encounter some other inconvenience, for example, the complexity and the uncertainty or vagueness. Fuzzy theory is considered as a more suitable setting for the sake of taking vagueness into consideration. Based on traditional cellular neural networks (CNNs), Yang and Yang proposed the fuzzy CNNs (FCNNs) [26], which integrates fuzzy logic into the structure of traditional CNNs and maintains local connectedness among cells. Unlike previous CNNs structures, FCNNs have fuzzy logic between its template input and/or output besides the sum of product operation. FCNNs are very useful paradigm for image processing problems, which is a cornerstone in image processing and pattern recognition. Therefor, it is necessary to consider both the fuzzy logic and delay effect on dynamical behaviors of neural networks. Nevertheless, to the best of our knowledge, there are few published papers considering the local robust stability of equilibrium points and domain of attraction for the fuzzy neural network (FNNs).

Therefore, in this paper, we will study the local robust stability of fuzzy neural network with time-varying and S-type distributed delays:

$$\begin{aligned}\dot{u}_i(t) = &- c_i(\lambda)u_i(t) + g_i\left(\sum_{j=1}^{n}\int_{-\tau(\lambda)}^{0} u_j(t+\theta)\,d\omega_{ij}(\theta,\lambda) + I_i\right) + \sum_{j=1}^{n} a_{ij}(\lambda) f_j(u_j(t)) \\ &+ \bigwedge_{j=1}^{n}\alpha_{ij}(\lambda) f_j(u_j(t-\tau_j(t))) + \bigvee_{j=1}^{n}\beta_{ij}(\lambda) f_j(u_j(t-\tau_j(t))), \quad i = 1,2,\ldots,n,\end{aligned} \tag{1.3}$$

where $\alpha_{ij}(\lambda)$ and $\beta_{ij}(\lambda)$ are elements of fuzzy feedback MIN template and fuzzy feedback MAX template, respectively. $a_{ij}(\lambda)$ are elements of feedback template. $u_i(t)$ stands for state of the ith neurons. $\tau_j(t)$ is the transmission delay and $f_j(t)$ is the activation function. $\wedge$ and $\vee$ denote the fuzzy AND and fuzzy OR operation, respectively. $\lambda \in \Xi \subset R$ is the parameter. The main purpose of this paper is to investigate local robust stability of equilibrium points of FNNs (1.3). Sufficient conditions are gained for local robust stability of equilibrium points. Meanwhile, the attracting domains of equilibrium points are also estimated.

Throughout this paper, we always assume the following

(A) $a_{ij}(\lambda), \alpha_{ij}(\lambda)$, and $\beta_{ij}(\lambda)$ are bounded in Ξ $(i, j = 1, 2, \ldots, n)$.

(H_1) $\inf_{\lambda\in\Xi} c_i(\lambda) > 0$, $0 \le \tau(\lambda) \le \tau$ and $\omega_{ij}(\theta, \lambda)$ $(i, j = 1, 2, \ldots, n)$ are nondecreasing bounded variation function on $[-\tau(\lambda), 0]$ with $\omega_{ij}(\theta, \lambda) > 0$, and $\int_{-\tau(\lambda)}^{0} u_j(t + \theta) d\omega_{ij}(\theta, \lambda)$ is Lebesgue-Stieltjes integral. $I = (I_1, I_2, \ldots, I_n)^T$ is a constant vector which denotes an external input.

(H_2) $g_i(\cdot)$, $i = 1, 2, \ldots, n$ are second-order differentiable, bounded, and Lipschitz continuous. There exist positive constants L_i and B_i such that $|g_i(x) - g_i(y)| \le L_i|x - y|$ and $|g_i(x)| \le B_i$ for any $x, y \in R$.

(H_3) The activation functions $f_i(u(t))$ with $f_i(0) = 0$ bounded and Lipschitz continuous; that is, there are some numbers $\mu_i > 0$ and $l_i > 0$ such that $|f_i(u)| \le \mu_i$ and $|f_i(u) - f_i(v)| \le l_i|u - v|$ for any $u, v \in R, i = 1, 2, \ldots, n$.

(H_4) Functions $\tau_j(t), j = 1, 2, \ldots, n$ are nonnegative, bounded, and continuously differentiable defined on R_+ and $0 \le \tau_j(t) \le \tau(\lambda)$.

The rest of this paper is organized as follows. In Section 2, we will give some basic definitions and basic results about the attracting domains of FNNs (1.3). In Section 3, we discuss the local robust stability of equilibrium points of FNNs (1.3). In Section 4, an example is given to illustrate the effectiveness of our results. Finally, we make a conclusion in Section 5.

2. Preliminaries

As usual, we denote by $C([-\tau(\lambda), 0], R^n)$ the set of all real-valued continuous mappings from $[-\tau(\lambda), 0]$ to R^n equipped with supremum norm $\|\cdot\|_\infty$ defined by

$$\|\phi\| = \max_{1\le i\le n} \sup_{-\tau(\lambda)<t\le 0} |\phi_i(t)|, \tag{2.1}$$

where $\phi = (\phi_1, \phi_2, \ldots, \phi_n)^T \in C([-\tau(\lambda), 0], R^n)$. Denote by $u(t, \phi, \lambda)$ the solution of FNNs (1.3) with initial condition $\phi \in C([-\tau(\lambda), 0], R^n)$.

Definition 2.1. A vector $u^*(\lambda) = (u_1^*(\lambda), u_2^*(\lambda), \ldots, u_n^*(\lambda))^T$ is said to be an equilibrium point of FNNs (1.3) if for each $i = 1, 2, \ldots, n$, one has

$$\begin{aligned} c_i(\lambda)u_i^*(\lambda) =& g_i\left(\sum_{j=1}^{n}\widetilde{\omega}_{ij}(\lambda)u_j^*(\lambda) + I_i\right) + \sum_{j=1}^{n} a_{ij}(\lambda) f_j\left(u_j^*(\lambda)\right) \\ &+ \bigwedge_{j=1}^{n}\alpha_{ij}(\lambda) f_j\left(u_j^*(\lambda)\right) + \bigvee_{j=1}^{n}\beta_{ij}(\lambda) f_j\left(u_j^*(\lambda)\right), \quad i = 1, 2, \ldots, n, \end{aligned} \tag{2.2}$$

where $\widetilde{\omega}_{ij}(\lambda) =: \int_{-\tau(\lambda)}^{0} d\omega_{ij}(\theta, \lambda)$. Denote by Ω the set of all equilibrium points of FNNs (1.3).

Definition 2.2. Let $u^*(\lambda) \in \Omega . u^*(\lambda)$ is said to be a locally robust attractive equilibrium point if for any given $\lambda \in \Xi$, there is a neighborhood $Y_\lambda(u^*(\lambda)) \subset C([-\tau(\lambda), 0], R^n)$ such that

$\phi \in \Upsilon_\lambda(u^*(\lambda))$ implies that $\lim_{t\to\infty}\|u(t,\phi,\lambda) - u^*(\lambda)\| = 0$. Otherwise, $u^*(\lambda)$ is said not to be a locally robust attractive equilibrium point. Denote by Ω_0 the set of all not locally robust attractive equilibrium points of FNNs (1.3).

Definition 2.3. Let $D, \widetilde{D}$ be subsets of R^n and let $u(t,\phi,\lambda)$ be a solution of FNNs (1.3) with $\phi \in C([-\tau(\lambda),0],\mathrm{R}^n)$.

(i) For any given $\lambda \in \Xi$, if $u(\sigma,\phi,\lambda) \in D$ for some $\sigma \geq 0$ implies that $u(t,\phi,\lambda) \in D$ for all $t \geq \sigma$, then D is said to be an attracting domain of FNNs (1.3).

(ii) For any given $\lambda \in \Xi$, if $\phi(\theta) \in \widetilde{D}$ for all $\theta \in [-\tau(\lambda),0]$ implies that $u(t,\phi,\lambda)$ converges to $u^*(\lambda)$, then $\widetilde{D}$ is said to be an attracting domain of $u^*(\lambda) \in \Omega$.

Correspondingly, the union of all attracting domains of equilibrium points of Ω is said to be an attracting domain of Ω.

For a class of differential equation with the term of fuzzy AND and fuzzy OR operation, there is the following useful inequality.

Lemma 2.4 ([26]). *Let $u = (u_1,u_2,\ldots,u_n)^T$ and $v = (v_1,v_2,\ldots,v_n)^T$ be two states of* (1.3)*; then one has*

$$\left|\bigwedge_{j=1}^{n}\alpha_{ij}f_j(u_j) - \bigwedge_{j=1}^{n}\alpha_{ij}f_j(v_j)\right| \leq \sum_{j=1}^{n}|\alpha_{ij}||f_j(u_j) - f_j(v_j)|,$$
$$\left|\bigvee_{j=1}^{n}\alpha_{ij}f_j(u_j) - \bigvee_{j=1}^{n}\alpha_{ij}f_j(v_j)\right| \leq \sum_{j=1}^{n}|\alpha_{ij}||f_j(u_j) - f_j(v_j)|. \tag{2.3}$$

Lemma 2.5. *Let $u(t)$ be any solution of FNNs* (1.3)*. Then $u(t)$ is uniformly bounded. Moreover, H is an attracting domain of FNNs* (1.3)*, where*

$$H =: H_1 \times H_2 \times \cdots \times H_n, \qquad H_i = \left[-\frac{B_i + M_i}{\inf_{\lambda\in\Xi}c_i(\lambda)}, \frac{B_i + M_i}{\inf_{\lambda\in\Xi}c_i(\lambda)}\right], \quad i = 1,2\ldots,n. \tag{2.4}$$

Proof. By (1.3) and Lemma 2.4, we have

$$\frac{\mathrm{d}^+}{\mathrm{d}t}|u_i(t)| \leq -\inf_{\lambda\in\Xi}c_i(\lambda)|u_i(t)| + B_i + M_i, \tag{2.5}$$

where

$$M_i = n\max_{1\leq j\leq n}\left\{\mu_j\sup_{\lambda\in\Xi}\left(\left|a_{ij}(\lambda)\right| + \left|\alpha_{ij}(\lambda)\right| + \left|\beta_{ij}(\lambda)\right|\right)\right\}, \quad i = 1,2\ldots,n. \tag{2.6}$$

By using differential inequality, we have for $t \geq \sigma$,

$$|u_i(t)| \leq \exp\left((\sigma - t)\inf_{\lambda\in\Xi}c_i(\lambda)\right)\left[|u_i(\sigma)| - \frac{B_i + M_i}{\inf_{\lambda\in\Xi}c_i(\lambda)}\right] + \frac{B_i + M_i}{\inf_{\lambda\in\Xi}c_i(\lambda)}, \quad i = 1,2\ldots,n, \tag{2.7}$$

which leads to the uniform boundedness of $u(t)$. Furthermore, given any $|u_i(\sigma)| \leq (B_i + M_i)/\inf_{\lambda\in\Xi} c_i(\lambda), i = 1, 2 \ldots, n$, we get for all $t \geq \sigma$,

$$|u_i(t)| \leq \frac{B_i + M_i}{\inf_{\lambda\in\Xi} c_i(\lambda)}. \tag{2.8}$$

Hence H is an attracting domain of FNNs (1.3). The proof is complete. □

By Lemma 2.4, we have the following theorem.

Theorem 2.6. *All equilibrium points of FNNs* (1.3) *lie in the attracting domain H, that is, $\Omega \subset H$.*

3. Local Robust Stability of Equilibrium Points

In this section, we should investigate local robust stability of equilibrium points of FNNs (1.3). We derive some sufficient conditions to guarantee local robust stable of equilibrium points in Ω/Ω_0 and estimate the attracting domains of these equilibrium points.

Theorem 3.1. *Let $u^*(\lambda) = (u_1^*(\lambda), u_2^*(\lambda), \ldots, u_n^*(\lambda))^T \in \Omega$. If there exist positive constants β_i ($i = 1, 2, \ldots, n$) such that for each $i = 1, 2, \ldots, n$*

$$\sum_{j=1}^{n} \beta_j \left(\sup_{\lambda\in\Xi} \{ \tilde{\omega}_{ji}(\lambda) | \dot{g}_j(\kappa_j(\lambda)) | + l_j (|a_{ij}(\lambda)| + |\alpha_{ij}(\lambda)| + |\beta_{ij}(\lambda)|) \} \right) < \beta_i \inf_{\lambda\in\Xi} c_i(\lambda), \tag{3.1}$$

where $\kappa_i(\lambda) = \sum_{j=1}^{n} \tilde{\omega}_{ij}(\lambda) u_j^*(\lambda) + I_i$, then one has the following.

(1) *$u^*(\lambda) \in \Omega/\Omega_0$, that is, $u^*(\lambda)$ is locally robust stable.*

(2) *Let*

$$\begin{aligned} \overline{R} =: 2 \min_{i\in N^+} \Bigg\{ & \frac{\beta_i \inf_{\lambda\in\Xi} c_i(\lambda)}{\sum_{k=1}^{n} \sum_{j=1}^{n} \beta_j \max_{\zeta\in R} |\ddot{g}_j(\zeta)| \sup_{\lambda\in\Xi} (\tilde{\omega}_{ji}(\lambda) \tilde{\omega}_{jk}(\lambda))} \\ & - \frac{\sum_{j=1}^{n} \beta_j (\sup_{\lambda\in\Xi} \{ \tilde{\omega}_{ji}(\lambda) | \dot{g}_j(\kappa_j(\lambda)) | + l_j (|a_{ij}(\lambda)| + |\alpha_{ij}(\lambda)| + |\beta_{ij}(\lambda)|) \})}{\sum_{k=1}^{n} \sum_{j=1}^{n} \beta_j \max_{\zeta\in R} |\ddot{g}_j(\zeta)| \sup_{\lambda\in\Xi} (\tilde{\omega}_{ji}(\lambda) \tilde{\omega}_{jk}(\lambda))} \Bigg\}. \end{aligned} \tag{3.2}$$

Then every solution $u(t, \phi, \lambda)$ of FNNs (1.3) *with $\phi \in O(u^*(\lambda))$ satisfies*

$$\lim_{t\to+\infty} \| u(t, \phi, \lambda) - u^*(\lambda) \|_\infty = 0, \tag{3.3}$$

where

$$O(u^*(\lambda)) = \left\{\phi \in C([-\tau(\lambda),0],R^n) : \|\phi - u^*(\lambda)\|_\infty < \frac{\overline{R}}{\sum_{i=1}^n(\beta_i/\min_{1\le i\le n}\beta_i)}\right\}. \quad (3.4)$$

(3) *The open set*

$$\bigcup_{u^*(\lambda)\in\Omega} B(u^*(\lambda)) =: \left\{u \in R^n : \|u - u^*(\lambda)\|_\infty < \frac{\overline{R}}{\sum_{i=1}^n(\beta_i/\min_{1\le i\le n}\beta_i)}\right\} \quad (3.5)$$

is an attracting domain of Ω, *and* $B(u^*(\lambda))$ *is an attracting domain of* $u^*(\lambda)$.

The proof of Theorem 3.1 relies on the following lemma.

Lemma 3.2. *Let* $u^*(\lambda) = (u_1^*(\lambda), u_2^*(\lambda), \dots, u_n^*(\lambda))^T \in \Omega$ *satisfying* (3.1). *Let* $u(t,\phi,\lambda)$ *be an arbitrary solution of FNNs* (1.3) *other than* u^*, *where* $\phi \in C([-\tau(\lambda),0],R^n)$. *Let*

$$V(t) = \sum_{i=1}^n \beta_i \left|u_i(t,\phi,\lambda) - u_i^*(\lambda)\right|, \quad (3.6)$$

where β_i *is given by* (3.1). *Then one has the following.*

(A_1) *If* $\|u_\sigma(\cdot,\phi,\lambda) - u^*(\lambda)\| < \overline{R}$ *for some* $\sigma \ge 0$, *then* $D^+V(\sigma) < 0$.

(A_2) *If* $\|\phi - u^*(\lambda)\|_\infty < \overline{R}/\sum_{i=1}^n(\beta_i/\min_{1\le i\le n}\beta_i)$ *and* $\sup_{\sigma-\tau\le s\le\sigma}V(s) \le \sup_{-\tau\le s\le 0}V(s)$ *for some* $\sigma \ge 0$, *then* $\|u_\sigma(\cdot,\phi,\lambda) - u^*(\lambda)\| < \overline{R}$.

(A_3) *If* $\|\phi - u^*(\lambda)\|_\infty < \overline{R}/(\sum_{i=1}^n \beta_i/\min_{1\le i\le n}\beta_i)$, *then* $D^+V(t) < 0$ *for all* $t \ge 0$.

Proof. Under transformation $y(t) = u(t,\phi,\lambda) - u^*(\lambda)$, we get that

$$\begin{aligned} \frac{d^+|y_i(t)|}{dt} \le{}& -c_i(\lambda)|y_i(t)| + \sum_{j=1}^n |a_{ij}(\lambda)||y_j(t)| + \sum_{j=1}^n(|\alpha_{ij}(\lambda)| + |\beta_{ij}(\lambda)|)|y_j(t-\tau_j(t))| \\ &+ \sum_{j=1}^n |\dot{g}_i(\kappa_i(\lambda))| \int_{-\tau(\lambda)}^0 |y_j(t+\theta)| d\omega_{ij}(\theta,\lambda) \\ &+ \frac{|\ddot{g}_i(\zeta_i)|}{2}\left(\sum_{j=1}^n \int_{-\tau(\lambda)}^0 |y_j(t+\theta)| d\omega_{ij}(\theta,\lambda)\right)^2, \end{aligned} \quad (3.7)$$

due to

$$
\begin{aligned}
&g_i\left(\sum_{j=1}^{n}\int_{-\tau(\lambda)}^{0}u_j(t+\theta)d\omega_{ij}(\theta,\lambda)+I_i\right)-g_i\left(\sum_{j=1}^{n}\int_{-\tau(\lambda)}^{0}u_j^*(\lambda)d\omega_{ij}(\theta,\lambda)+I_i\right)\\
&=\dot{g}_i\left(\sum_{j=1}^{n}\widetilde{\omega}_{ij}(\lambda)u_j^*(\lambda)+I_i\right)\sum_{j=1}^{n}\int_{-\tau(\lambda)}^{0}\left|y_j(t+\theta)\right|d\omega_{ij}(\theta,\lambda)\\
&\quad+\frac{\left|\ddot{g}_i(\zeta_i)\right|}{2}\left(\sum_{j=1}^{n}\int_{-\tau(\lambda)}^{0}\left|y_j(t+\theta)\right|d\omega_{ij}(\theta,\lambda)\right)^2,
\end{aligned}
\tag{3.8}
$$

where ζ_i lies between $\sum_{j=1}^{n}\int_{-\tau(\lambda)}^{0}u_j(t+\theta)d\omega_{ij}(\theta,\lambda)+I_i$ and $\sum_{j=1}^{n}\int_{-\tau(\lambda)}^{0}u_j^*(\lambda)d\omega_{ij}(\theta,\lambda)+I_i$. From (3.7), we can derive that

$$
\begin{aligned}
\frac{d^+V(t)}{dt}\le{}&\sum_{i=1}^{n}\beta_i\Bigg\{-\inf_{\lambda\in\Xi}c_i(\lambda)\left|y_i(t)\right|+\sum_{j=1}^{n}l_j\left|a_{ij}(\lambda)\right|\left|y_j(t)\right|\\
&\quad+\sum_{j=1}^{n}l_j\left(\left|\alpha_{ij}(\lambda)\right|+\left|\beta_{ij}(\lambda)\right|\right)\left|y_j(t-\tau_j(t))\right|\\
&\quad+\left[\left|\dot{g}_i(\kappa_i(\lambda))\right|+\frac{\left|\ddot{g}_i(\zeta_i)\right|}{2}\sum_{j=1}^{n}\int_{-\tau(\lambda)}^{0}\left|y_j(t+\theta)\right|d\omega_{ij}(\theta,\lambda)\right]\\
&\quad\times\sum_{j=1}^{n}\int_{-\tau(\lambda)}^{0}\left|y_j(t+\theta)\right|d\omega_{ij}(\theta,\lambda)\Bigg\}\\
\le{}&\sum_{i=1}^{n}\beta_i\Bigg\{-\inf_{\lambda\in\Xi}c_i(\lambda)\left|y_i(t)\right|+\sum_{j=1}^{n}l_j\sup_{\lambda\in\Xi}\left(\left|a_{ij}(\lambda)\right|+\left|\alpha_{ij}(\lambda)\right|+\left|\beta_{ij}(\lambda)\right|\right)\sup_{t-\tau\le s\le t}\left|y_j(s)\right|\\
&\quad+\left[\left|\dot{g}_i(\kappa_i(\lambda))\right|+\frac{\left|\ddot{g}_i(\zeta_i)\right|}{2}\sum_{j=1}^{n}\widetilde{\omega}_{ij}(\lambda)\sup_{t-\tau\le s\le t}\left|y_j(s)\right|\right]\times\sum_{j=1}^{n}\widetilde{\omega}_{ij}(\lambda)\sup_{t-\tau\le s\le t}\left|y_j(s)\right|\Bigg\}\\
\le{}&\sum_{i=1}^{n}\beta_{\mathrm{i}}\Bigg\{-\inf_{\lambda\in\Xi}c_i(\lambda)\left|y_i(t)\right|+\sum_{j=1}^{n}l_j\sup_{\lambda\in\Xi}\left(\left|a_{ij}(\lambda)\right|+\left|\alpha_{ij}(\lambda)\right|+\left|\beta_{ij}(\lambda)\right|\right)\sup_{t-\tau\le s\le t}\left|y_j(s)\right|\\
&\quad+\left[\left|\dot{g}_i(\kappa_i(\lambda))\right|+\frac{\left|\ddot{g}_i(\zeta_i)\right|}{2}\sum_{j=1}^{n}\widetilde{\omega}_{ij}(\lambda)\sup_{t-\tau\le s\le t}\left|y_j(s)\right|\right]\times\sum_{j=1}^{n}\widetilde{\omega}_{ij}(\lambda)\sup_{t-\tau\le s\le t}\left|y_j(s)\right|\Bigg\}
\end{aligned}
$$

$$\le \sum_{i=1}^{n}\left\{-\beta_i \inf_{\lambda\in\Xi} c_i(\lambda) + \sum_{j=1}^{n}\beta_j l_j \sup_{\lambda\in\Xi}\left(\left|a_{ij}(\lambda)\right| + \left|\alpha_{ij}(\lambda)\right| + \left|\beta_{ij}(\lambda)\right|\right)\right.$$
$$\left. + \sum_{j=1}^{n}\beta_j \widetilde{\omega}_{ji}(\lambda)\left[\left|\dot{g}_j(\kappa_j(\lambda))\right| + \frac{\left|\ddot{g}_j(\zeta_j)\right|}{2}\sum_{j=1}^{n}\widetilde{\omega}_{jk}(\lambda)\sup_{t-\tau\le s\le t}\left|y_k(s)\right|\right]\right\}\sup_{t-\tau\le s\le t}\left|y_i(s)\right|. \tag{3.9}$$

As $\|u_\sigma(\cdot,\phi,\lambda) - u^*(\lambda)\|_\infty < \overline{R}$, we have for each $i = 1,2,\dots,n, \sup_{t-\tau\le s\le t}|y_i(s)| < \overline{R}$, which imply that $D^+V(\sigma) < 0$.

Since $\min_{1\le i\le n}\{\beta_i\}\|u_\sigma(\cdot,\phi,\lambda) - u^*(\lambda)\|_\infty \le \sup_{\sigma-\tau\le s\le\tau} V(s)$ and

$$\sup_{-\tau\le s\le 0} V(s) = \sum_{i=1}^{n}\beta_i\left\{\sup_{-\tau\le s\le 0}\left|u_i(s,\phi,\lambda) - u_i^*(\lambda)\right|\right\} \le \sum_{i=1}^{n}\beta_i\|\phi - u^*(\lambda)\|_\infty, \tag{3.10}$$

we have $\|u_\sigma(\cdot,\phi,\lambda) - u^*(\lambda)\|_\infty \le \sum_{i=1}^{n}(\beta_i/\min_{1\le i\le n}\beta_i)\|\phi - u^*(\lambda)\|_\infty < \overline{R}$.

Since $\|\phi - u^*(\lambda)\|_\infty < \overline{R}/\sum_{i=1}^{n}(\beta_i/\min_{1\le i\le n}\beta_i) < \overline{R}$, from (A_1), we know that $D^+V(0) < 0$. We assert that (A_3) holds. Otherwise, there exist $t_0 > 0$ such that $D^+V(t_0) \ge 0$ and $D^+V(t) < 0$ for all $t \in [0,t_0)$. This implies that $V(t)$ is strictly monotonically decreasing on the interval $[0,t_0]$. It is obvious that $\sup_{t_0-\tau\le s\le t_0} V(s) \le \sup_{-\tau\le s\le t_0} V(s)$. By using (A_2), we get that $\|u_{t_0}(\cdot,\phi,\lambda) - u^*(\lambda)\|_\infty < \overline{R}$. From $(A_1), D^+V(t_0) < 0$. This leads to a contradiction. Hence $D^+V(t) < 0$ for all $t \ge 0$. □

Now we are in a position to complete the proof of Theorem 3.1.

Proof. Let $u(t,\phi,\lambda)$ be an arbitrary solution of FNNs (1.3) other than $u^*(\lambda)$ and satisfy $\|\phi - u^*(\lambda)\|_\infty < \overline{R}/\sum_{i=1}^{n}(\beta_i/\min_{1\le i\le n}\beta_i)$. It follows from (A_3) that $D^+V(t) < 0$ for all $t \ge 0$, that is, $\sup_{t-\tau\le s\le t} V(s) \le \sup_{-\tau\le s\le 0} V(s)$ for all $t \ge 0$. Together with (A_2) we get $\|u_t(\cdot,\phi,\lambda) - u^*(\lambda)\|_\infty < \overline{R}$ for all $t \ge 0$. Take

$$\begin{aligned}
\chi_i &= \beta_i \inf_{\lambda\in\Xi} c_i(\lambda) \\
&\quad - \sum_{j=1}^{n}\beta_j\left(\sup_{\lambda\in\Xi}\left\{\widetilde{\omega}_{ji}(\lambda)\left|\dot{g}_j(\kappa_j(\lambda))\right|\right\} + l_j \sup_{\lambda\in\Xi}\left(\left|a_{ij}(\lambda)\right| + \left|\alpha_{ij}(\lambda)\right| + \left|\beta_{ij}(\lambda)\right|\right)\right), \\
\eta_i &= \sum_{k=1}^{n}\sum_{j=1}^{n}\beta_j \max_{\zeta\in R}\left|\ddot{g}_j(\zeta)\right| \sup_{\lambda\in\Xi}\left(\widetilde{\omega}_{ji}(\lambda)\widetilde{\omega}_{jk}(\lambda)\right)\overline{R}.
\end{aligned} \tag{3.11}$$

It is obvious that $\chi_i - \eta_i > 0$ for each $i = 1,2,\dots,n$. From (3.9) we have

$$D^+V(t) < -\min_{1\le i\le n}\{\chi_i - \eta_i\}\sum_{i=1}^{n}\sup_{t-\tau\le s\le t}\left|y_i(s)\right| \le -\min_{1\le i\le n}\{\chi_i - \eta_i\}\sum_{i=1}^{n}\left|y_i(s)\right|. \tag{3.12}$$

By integrating both sides of above inequality from 0 to t, we have

$$V(t) + \min_{1\le i\le n}\{\chi_i - \eta_i\}\int_0^t \sum_{i=1}^n |y_i(s)| \le V(0). \tag{3.13}$$

It follows that

$$\limsup_{t\to\infty} \min_{1\le i\le n}\{\chi_i - \eta_i\}\int_0^t \sum_{i=1}^n |y_i(s)| \le V(0) < \infty. \tag{3.14}$$

Note that $u(t,\phi,\lambda)$ is bounded on R_+ by Lemma 2.4; it follows from FNNs (1.3) that $\dot{u}$ is bounded on R_+. Hence $|u(t,\phi,\lambda) - u^*(\lambda)|$ is uniformly continuous on R_+. From Lemma 2.5, we get that $\lim_{t\to\infty}\sum_{i=1}^n |u_i(t,\phi,\lambda) - u_i^*(\lambda)| = 0$. So the assertions of (1) and (2) hold. Let us consider an arbitrary solution $u(t,\phi,\lambda)$ of FNNs (1.3) satisfying $\phi(s) \in B(u^*(\lambda))$ for all $s \in [-\tau(\lambda), 0]$ and some $u^*(\lambda) \in \Omega$. Then it is obvious that

$$\|\phi - u^*(\lambda)\|_\infty < \frac{\overline{R}}{\sum_{i=1}^n \beta_i / \min_{1\le i\le n}\beta_i}. \tag{3.15}$$

From (2), we get $\lim_{t\to\infty}\|u(t,\phi,\lambda) - u^*(\lambda)\|_\infty = 0$. Hence $B(u^*(\lambda))$ is an attracting domain of $u^*(\lambda)$. Consequently, the open set $\cup_{u^*(\lambda)\in\Omega} B(u^*(\lambda))$ is an attracting domain of Ω. The proof is complete. □

Corollary 3.3. *Let $u^*(\lambda) = (u_1^*(\lambda), u_2^*(\lambda), \ldots, u_n^*(\lambda))^T \in \Omega$. If there exist positive constants β_i $(i = 1,2,\ldots,n)$ such that for each $i = 1,2,\ldots,n$*

$$\sum_{j=1}^n \beta_j \left(\{\tilde{\omega}_{ji}|\dot{g}_j(\kappa_j)|\} + l_j \sup_{\lambda\in\Xi}(|a_{ij}(\lambda)| + |\alpha_{ij}(\lambda)| + |\beta_{ij}(\lambda)|) \right) < \beta_i c_i, \tag{3.16}$$

where $\kappa_i = \sum_{j=1}^n \tilde{\omega}_{ij} u_j^*(\lambda) + I_i$, then one has the following.

(1) *$u^*(\lambda) \in \Omega/\Omega_0$, that is, $u^*(\lambda)$ is locally asymptotically stable.*

(2) *Let*

$$\begin{aligned}\overline{R} =: 2\min_{i\in N^+}\Bigg\{ &\frac{\beta_i c_i}{\sum_{k=1}^n \sum_{j=1}^n \beta_j \max_{\zeta\in R}|\ddot{g}_j(\zeta)|\tilde{\omega}_{ji}\tilde{\omega}_{jk}} \\ &- \frac{\sum_{j=1}^n \beta_j (\{\tilde{\omega}_{ji}|\dot{g}_j(\kappa_j)|\} + l_j \sup_{\lambda\in\Xi}(|a_{ij}(\lambda)| + |\alpha_{ij}(\lambda)| + |\beta_{ij}(\lambda)|))}{\sum_{k=1}^n \sum_{j=1}^n \beta_j \max_{\zeta\in R}|\ddot{g}_j(\zeta)|\tilde{\omega}_{ji}\tilde{\omega}_{jk}} \Bigg\}.\end{aligned} \tag{3.17}$$

Then every solution $u(t,\phi,\lambda)$ of FNNs (1.3) *with $\phi \in O(u^*(\lambda))$ satisfies*

$$\lim_{t\to+\infty}\|u(t,\phi,\lambda) - u^*(\lambda)\|_\infty = 0, \tag{3.18}$$

where

$$O(u^*(\lambda)) = \left\{\phi \in C([-\tau,0],R^n) : \|\phi - u^*(\lambda)\|_\infty < \frac{\overline{R}}{\sum_{i=1}^n (\beta_i/\min_{1\le i\le n}\beta_i)}\right\}. \tag{3.19}$$

(3) *The open set*

$$\bigcup_{u^*(\lambda)\in\Omega} B(u^*(\lambda)) =: \left\{u \in R^n : \|u - u^*(\lambda)\|_\infty < \frac{\overline{R}}{\sum_{i=1}^n (\beta_i/\min_{1\le i\le n}\beta_i)}\right\} \tag{3.20}$$

is an attracting domain of Ω, *and* $B(u^*(\lambda))$ *is an attracting domain of* $u^*(\lambda)$.

4. Illustrative Example

For convenience of illustrative purpose, we only consider simple fuzzy neural network with time-varying and S-type distributed delays satisfying

$$\omega_{ij}(\theta,\lambda) = \begin{cases} \omega_{ij}(\lambda), & \theta = 0, \\ & \qquad \tau(\lambda) \equiv \tau. \\ 0, & -\tau \le \theta < 0, \end{cases} \tag{4.1}$$

Then fuzzy neural network with two neurons can be modeled by

$$\begin{aligned} \dot{u}_i(t) = &- c_i(\lambda)u_i(t) + g_i\left(\sum_{j=1}^2 u_j(t)\omega_{ij}(\lambda) + I_i\right) + \sum_{j=1}^2 a_{ij}(\lambda)f_j(u_j(t)) \\ &+ \bigwedge_{j=1}^2 \alpha_{ij}(\lambda)f_j(u_j(t-\tau_j(t))) + \bigvee_{j=1}^2 \beta_{ij}(\lambda)f_j(u_j(t-\tau_j(t))), \quad i = 1,2. \end{aligned} \tag{4.2}$$

Take

$$\begin{aligned} &c_1(\lambda) = \tanh(4-2\sin\lambda), \quad \omega_{11}(\lambda) = 4.02 - 2\sin\lambda, \quad \omega_{12}(\lambda) = 0.02, \\ &c_2(\lambda) = \tanh(2.3-\cos\lambda), \quad \omega_{21}(\lambda) = 0.01, \quad \omega_{22}(\lambda) = 2.31 - \cos\lambda, \\ &g_1(\xi) = g_2(\xi) = \tanh\xi, \quad \Xi = \left[0, \frac{\pi}{2}\right], \quad I_1 = -0.02, \quad I_2 = -0.01, \\ &\qquad \tau_j(t) = \tau\arctan\frac{2}{\pi}t, \quad j = 1,2, \\ &f_1(\xi) = f_2(\xi) = \sin\pi\xi, \quad a_{ij}(\lambda) = \alpha_{ij}(\lambda) = \beta_{ij}(\lambda) = \frac{-\sin\lambda}{100}, \quad i,j = 1,2. \end{aligned} \tag{4.3}$$

It is easy to check that (H_1)–(H_5) hold and $L_i = B_i = \mu_i = l_i = 1$ for $i = 1, 2$. We can check that

$$\sum_{i=1}^{2}\left(\sup_{\lambda\in[0,\pi/2]}\omega_{i1}(\lambda)L_i + l_1\sup_{\lambda\in[0,\pi/2]}\left(|a_{i1}(\lambda)| + |\alpha_{i1}(\lambda)| + \left|\beta_{i1}(\lambda)\right|\right)\right) = 4.1 > \inf_{\lambda\in[0,\pi/2]}c_1(\lambda) = \tanh 2. \tag{4.4}$$

From simple calculations, we know that $[-1.06/\tanh 2, 1.06/\tanh 2] \times [-1.06/\tanh 1.3, 1.06/\tanh 1.3]$ is an attracting domain of FNNs (4.2). All equilibrium points of FNNs (4.2) lie in $[-1.06/\tanh 2, 1.06/\tanh 2] \times [-1.06/\tanh 1.3, 1.06/\tanh 1.3]$. From some calculations, we have two equilibrium points $O_1 = (1, 0), O_2 = (0, 1)$. For equilibrium $O_2 = (1, 1)$, we have $\kappa_1(\lambda) = 4 - 2\sin\lambda, \kappa_2(\lambda) = 2.3 - \cos\lambda$ and $\sup_{\lambda\in[0,\pi/2]}|\dot{g}_1(\kappa_1(\lambda))| = 0.0680, \sup_{\lambda\in[0,\pi/2]}|\dot{g}_2(\kappa_2(\lambda))| = 0.0386$. Taking $\beta_1 = \beta_2 = 1$, we get

$$\begin{aligned}
&\sum_{j=1}^{2}\left(\sup_{\lambda\in[0,\pi/2]}\omega_{1j}(\lambda)\left|\dot{g}_j(\kappa_j(\lambda))\right| + l_j\sup_{\lambda\in[0,\pi/2]}\left(\left|a_{1j}(\lambda)\right| + \left|\alpha_{1j}(\lambda)\right| + \left|\beta_{1j}(\lambda)\right|\right)\right)\\
&\qquad < 0.04 < \tanh 2 = \inf_{\lambda\in[0,\pi/2]}c_1(\lambda),\\
&\sum_{j=1}^{2}\left(\sup_{\lambda\in[0,\pi/2]}\omega_{2j}(\lambda)\left|\dot{g}_j(\kappa_j(\lambda))\right| + l_j\sup_{\lambda\in[0,\pi/2]}\left(\left|a_{2j}(\lambda)\right| + \left|\alpha_{2j}(\lambda)\right| + \left|\beta_{2j}(\lambda)\right|\right)\right)\\
&\qquad < 0.04 < \tanh 1.3 = \inf_{\lambda\in[0,\pi/2]}c_2(\lambda).
\end{aligned} \tag{4.5}$$

Similarly, we can check that (3.1) holds for O_k $(k = 1, 2)$. Therefore, from Theorem 3.1, the four equilibrium points O_k $(k = 1, 2)$ are locally robust stable and their convergent radius is 0.04.

Remark 4.1. The above example implies that the system has multiple equilibrium points under the (relevant) assumption of monotone nondecreasing activation functions. These equilibrium points do not globally converge to the unique equilibrium point.

5. Conclusions

In this paper, we derive some sufficient conditions for local robust stability of fuzzy neural network with time-varying and S-type distributed delays and give an estimate of attracting domains of stable equilibrium points except isolated equilibrium points. Our results not only show local robust stability of equilibrium points but also allow much broader application for fuzzy neural network with or without delays. An example is given to show the effectiveness of our results.

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